GENERIC ELASTIC MEDIA.

PER HOLM

Institute of Mathematics, University of Oslo.

To Jens Lothe on the occation of his sixtieth birthday

ABSTRACT. We introduce the class of generic elstic media as those having stable acoustic axes and set up the framework for a topological study of the polarization fields of elastic waves in anisotropic solids. We show that generic media must have an even number of accoustic axes, at most sixteen. The number of axes is invariant under small perturbations of the elasticity coefficients. The class of generic media is separated into open subsets by a (nongeneric) discriminant hypersurface in the space of all elastic media.

In recent years topological methods have come to play a conspicuous role in different parts of physics. The purpose of this paper is to provide a suitable background for a topological study of the polarization fields of elastic waves in anisotropic media. In [3] such a study was initiated for media without acoustic axes. Here we consider the more general situation of media with stable acoustic axes. (See [2] for an illuminating introduction to elastic waves in anisotropic solids and [1] for the more specific study of acoustic axes.)

1. OVERVIEW. STATEMENTS. To set notation we briefly recall the basics. The propagation of elastic waves in a homogenous solid is governed by a system of three linear second order partial differential equations with constant coefficients:

$$\frac{\partial^2 u_p}{\partial t^2} - \sum_{q,r,s} C_{pq,rs} \frac{\partial^2 u_r}{\partial x_q \partial x_s} = 0 \qquad (p = 1, 2, 3) \tag{1}$$

The elastic constants $C_{pq,rs}$ are the components of a fourth order tensor C with symmetry properties

$$C_{pq,rs} = C_{qp,rs} = C_{pq,sr} = C_{rs,pq} \tag{2}$$

and a positivity property

$$\sum_{p,q,r,s} C_{pq,rs} y_{pq} y_{rs} > 0 \tag{3}$$

for any non-zero symmetric matrix $Y = (y_{rs})$ (where the indices run from 1 to 3).

Consider a monocromatic plane wave solution of the system (1),

$$\mathbf{u} = \mathbf{a} e^{i(\mathbf{x}\cdot\mathbf{y} - \omega t)} \tag{4}$$

Here $\mathbf{a}$ is the polarisation and $\mathbf{y}$ is the wave vector. We regard $\mathbf{u}$ as a function of t and $\mathbf{x} = (x_1, x_2, x_3)$, depending on additional parameters ω, $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$.

These parameters are not independent. Inserting the expression (4) for $\mathbf{u}$ in (1) we obtain the conditions

$$\omega^2 a_p - \sum_{q,r,s} C_{pq,rs} y_q y_s a_r = 0 \qquad (p = 1, 2, 3)$$

or more succinctly

$$(\omega^2 E - Y'')\mathbf{a} = \mathbf{o}, \tag{5}$$

Y'' being the Christoffel matrix of C, with entries $y''_{pr} = \sum_{q,s} C_{pq,rs} y_q y_s$, and E the identity matrix. Thus $\mathbf{a}$ and ω depend algebraically on $\mathbf{y}$ according to the variable eigenvalue equation (5). The correspondence $\mathbf{y} \mapsto Y''$ will be called the Christoffel mapping (of C).

It follows from (2) and (3) that Y'' is a positive definite symmetric matrix for any $\mathbf{y} \neq \mathbf{o}$. Moreover the Christoffel mapping is an even function of the wave vector, i.e. $Y''(-\mathbf{y}) = Y''(\mathbf{y})$, since the entries of Y'' are quadratic forms in the components of $\mathbf{y}$. On the unit sphere $|\mathbf{y}| = 1$ the frequency ω coincides with the phase velocity c. Consequently, for each (unoriented) direction $\pm\mathbf{y}$ there are typically three sound waves with different velocities c_1, c_2, c_3 and mutually orthogonal polarizations $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. An acoustic axis is a direction for which two sound velocities coincide.

In the sequal we will consider the elasticity tensors as linear operators on the inner product space $\mathbf{M}$ of real 3×3-matrices. In fact the operation $Y \mapsto Y'$ where

$$y'_{pq} = \sum_{r,s} C_{pq,rs} y_{rs},$$

identifies $C = (C_{pq,rs})$ as the standard matrix of a symmetric operator on $\mathbf{M}$, also denoted C. Precisely C maps $\mathbf{S}$ into $\mathbf{S}$ and $\mathbf{A}$ into $\mathbf{A}$, and is moreover zero on $\mathbf{A}$. Here $\mathbf{M} = \mathbf{S} + \mathbf{A}$ is the direct sum decomposition of $\mathbf{M}$ defined by the splitting of a matrix into its symmetric- and anti-symmetric part. Thus condition (2) identifies that space of tensors with $\mathcal{S} = \mathrm{Sym}(\mathbf{S})$, the space of symmetric operators on $\mathbf{S}$, while condition (3) picks out the subset $\mathcal{P}$ of positive definite operators in $\mathcal{S}$. Quite generally we will not distinguish between a (general) tensor $T = (T_{pq,rs})$ and the linear operator on $\mathbf{M}$ it defines in this way.

On the other hand $T = (T_{pq,rs})$ also acts on $\mathbf{M}$ according to the rule

$$y''_{pr} = \sum_{qs} T_{pq,rs} y_{qs}.$$

This yields a linear operator which we denote $\hat{T}$. If T comes from $\mathcal{P}$ (or just $\mathcal{S}$), the operator $\hat{T}$ is symmetric and *split* (i.e. preserves the decomposition $\mathbf{M} = \mathbf{S} + \mathbf{A}$). But it need not vanish on $\mathbf{A}$, nor be positive definit on $\mathbf{S}$. In fact it can be singular on $\mathbf{S}$; how singular is an interesting question.

The correspondence $T \mapsto \hat{T}$ is a linear involution on the algebra of operators on $\mathbf{M}$. Therefore, as C runs through $\mathcal{S}$, $\hat{C}$ runs through a linear subspace $\hat{\mathcal{S}}$ of $\mathrm{Lin}(\mathbf{M})$. And as C runs through $\mathcal{P}$, $\hat{C}$ runs through the corresponding subset $\hat{\mathcal{P}}$. Set $\hat{\mathcal{S}} = \mathcal{E}$ and $\hat{\mathcal{P}} = \mathcal{Q}$.

The operator $\hat{C} \in \mathcal{E}$ maps a matrix Y with entries $y_{qs} = y_q y_s$ to the matrix Y'' with entries $y''_{pr} = \sum_{qs} C_{pq,rs} y_q y_s$. In other words the Christoffel mapping can be viewed as a composition; the quadratic mapping $Y = Y(\mathbf{y})$ followed by the linear $Y'' = \hat{C}(Y)$. And the direction of $\mathbf{y} = (y_1, y_2, y_3)$ is an accoustic axis for C if Y'' has multiple eigenvalues. We will say that $\pm\mathbf{y}$ is a *stable* accoustic axis if $\mathbf{R}^3$ is mapped transversely to Δ at $\mathbf{y}$, where Δ is the subvariety of matrices in $\mathbf{S}$ with multiple eigenvalues. We call C *generic* if it has only stable accoustic axes. (But see the remark at the end of section 6 for a more comprehensive definition.)

Thus the behaviour of the acoustic axes discriminates between the elasticity operators and hence between elastic media. More precisely there is a closed projective semialgebraic set $\mathcal{D}$ of codimension 1 in $\mathcal{S}$ (hence of dimension 20) representing media with unstable acoustic axes. Moreover $\mathcal{D}$ is filtered into lower dimensional sets representing media of higher degeneracy. Restricting to the positivity cone $\mathcal{P}$ we get the corresponding filtration of $\mathcal{P}$. Of course the stratum $\mathcal{S} - \mathcal{D}$ is an open dense set in $\mathcal{S}$, and so the same is true for $\mathcal{P} - \mathcal{D}$ in $\mathcal{P}$.

Our main concern will be the generic stratum $\mathcal{S}-\mathcal{D}$, or rather $\mathcal{P}-\mathcal{D}$. The set $\mathcal{S}-\mathcal{D}$ is subdivided by the hypersurface $\mathcal{D}$ into nine disjoint open subsets $\mathcal{S}_0, \mathcal{S}_2, \ldots, \mathcal{S}_{16}$ and the set $\mathcal{P}-\mathcal{D}$ correspondingly into $\mathcal{P}_0$, $\mathcal{P}_2, \ldots, \mathcal{P}_{16}$, each specified by the number of accoustic axes of its members. In fact a generic medium must have an even number of accoustic axes, at most sixteen. The number of axes is then invariant under small perturbations of the elasticity coefficients.

Each genericity class $\mathcal{P}_{2i}$ is possibly further subdivided into finitely many components containing those stable media that can be stably deformed to each other by a one-parameter deformation of the coefficients. Thus the problem arizes of determining the components of any given genericity class, say by constructing a "prototype" polarization frame field for each component. This problem was solved for the class $\mathcal{P}_0$ in [3], where it was shown that $\mathcal{P}_0$ is connected and admits a constant frame field as prototype. A similarly simple situation is not likely in the general case.

Incidentally, note that perturbations means general (e.g. triclinic) perturbations. Media with additional symmetry are likely to be exceptional (i.e. belong to $\mathcal{D}$) rather than generic in this setup. To deal with media with specific symmetries one should to start with the correspondingly reduced moduli spaces $\mathcal{S}'$ and $\mathcal{P}'$ and then proceed much as below.

Finally we remark that the discriminant set $\mathcal{D} \subset \mathcal{S}$ appears most naturally as $\hat{\mathcal{D}} \subset \mathcal{E}$. At any rate that is how it is introduced below.

2. THE ELASTICITY OPERATORS. We denote by $\mathbf{M}$ the algebra of real 3-by-3 matrices and by $\mathbf{S}$ and $\mathbf{A}$ the linear subspaces of symmetric and antisymmetric matrices, of dimensions 6 and 3, respectively. As a linear space $\mathbf{M}$ is the direct sum of $\mathbf{S}$ and $\mathbf{A}$. We shall give $\mathbf{M}$ the inner product $\langle Y, Z\rangle = \mathrm{tr}(Y^tZ) = \sum_{i,j} y_{ij}z_{ij}$. Then $\mathbf{S}$ and $\mathbf{A}$ become orthogonal complements.

Since $\mathbf{M}$ is a linear space, we may consider $\mathrm{Lin}(\mathbf{M})$, the algebra of linear operators on $\mathbf{M}$, as well as some useful subsets. First there is $\mathrm{Split}(\mathbf{M})$, the linear subspace of split operators. As mentioned we call an operator split if it maps $\mathbf{A}$ into $\mathbf{A}$ and $\mathbf{S}$ into $\mathbf{S}$. Evidently $\mathrm{Split}(\mathbf{M})$ is the direct sum of $\mathrm{Lin}(\mathbf{A})$ and $\mathrm{Lin}(\mathbf{S})$, provided we identify $\mathrm{Lin}(\mathbf{A})$ with the subspace of split operators on $\mathbf{M}$ vanishing on $\mathbf{S}$ and $\mathrm{Lin}(\mathbf{S})$ with the subspace of split operators on $\mathbf{M}$ vanishing on $\mathbf{A}$. In fact $\mathrm{Split}(\mathbf{M})$ is a subalgebra of $\mathrm{Lin}(\mathbf{M})$, and $\mathrm{Lin}(\mathbf{A})$ and $\mathrm{Lin}(\mathbf{S})$ are ideals of $\mathrm{Split}(\mathbf{M})$. A simple computation shows that $\mathrm{Split}(\mathbf{M})$, $\mathrm{Lin}(\mathbf{A})$ and $\mathrm{Lin}(\mathbf{S})$ have dimensions 45, 9 and 36, respectively.

Next there is $\mathrm{Sym}(\mathbf{M})$, the linear subspace of symmetric operators on $\mathbf{M}$. It meets Split($\mathbf{M}$) in Splitsym($\mathbf{M}$), the subspace of split symmetric operators on $\mathbf{M}$. It is clear that Splitsym($\mathbf{M}$) is the direct sum of Sym($\mathbf{A}$) and Sym($\mathbf{S}$) under the identifications above.

If T is a linear operator on $\mathbf{M}$ with matrix $(T_{pq,rs})$ relative to the standard basis (E_{ij}), we write $T = (T_{pq,rs})$. Then $T_{pq,rs} = \langle E_{pq}, T(E_{rs})\rangle$, since (E_{ij}) is in fact orthonormal. Thus composition of operators is translated into matrix multiplication, so that the equation $T = R \circ S$ in Lin($\mathbf{M}$) is equivalent to $T_{pq,tu} = \sum_{r,s} R_{pq,rs} S_{rs,tu}$. Moreover the value $Y' = T(Y)$ of an operator T on a matrix Y in $\mathbf{M}$ is given by

$$y'_{pq} = \sum_{r,s} T_{pq,rs} y_{rs} \tag{6}$$

It is usefull to have the different operator types characterized by conditions on their matrix entries. For Split($\mathbf{M}$) and Sym($\mathbf{M}$) the conditions are respectively

$$\begin{aligned} T_{pq,rs} &= T_{qp,sr} \\ T_{pq,rs} &= T_{rs,pq} \end{aligned}$$

while for Splitsym($\mathbf{M}$), Sym($\mathbf{S}$) and Sym($\mathbf{A}$) they are

$$\begin{aligned} T_{pq,rs} &= T_{rs,pq} = T_{qp,sr} \\ T_{pq,rs} &= T_{rs,pq} = T_{qp,rs} = T_{pq,sr} \\ T_{pq,rs} &= T_{rs,pq} = -T_{qp,rs} = -T_{pq,sr}. \end{aligned}$$

In this way the linear space all of fourth order tensors on $\mathbf{R}^3$ is identified with $\mathcal{L} = \text{Lin}(\mathbf{M})$ and the subspace of tensors satisfying (2) with $\mathcal{S} = \text{Sym}(\mathbf{S})$. Moreover the set of tensors also satisfying the inequalities (3) is identified with the set $\mathcal{P}$ of positive definite operators in Sym($\mathbf{S}$).

From the point of view of elasticity theory the relevant operation of a tensor $(T_{pq,rs})$ is given not so much by (6) as by

$$y''_{pq} = \sum_{r,s} T_{pr,qs} y_{rs} \tag{7}$$

This defines a different linear operator on $\mathbf{M}$ which we denote $\hat{T}$. The matrix of $\hat{T}$ is of course given by

$$\hat{T}_{pq,rs} = T_{pr,qs} \tag{8}$$

and we can write the relation (7) as $Y'' = \hat{T}(Y)$. Clearly $\hat{}$ is an involution on $\mathcal{L}$, i.e. $\hat{}$ is linear and of order two.

Next observe that Split(**M**) and Sym(**M**) are interchanged by the twisting $\hat{}$. Since $\hat{}$ is of order 2 and both spaces are of the same dimension, it suffices to check that Sym(**M**) is mapped into Split(**M**). But if T is symmetric, then $\hat{T}_{pq,rs} = T_{pr,qs} = T_{qs,pr} = \hat{T}_{qp,sr}$, showing that $\hat{T}$ is split.

It follows that the space of split symmetric operators on **M** is in fact invariant under the twisting, so that $\hat{}$ by restriction is an involution on Splitsym(**M**). On the other hand the splitting subspaces Sym(**A**) and Sym(**S**) are not themselves invariant, so that Splitsym(**M**) receives a new splitting Splitsym(**M**) $= \widehat{\mathrm{Sym}}(\mathbf{A}) + \widehat{\mathrm{Sym}}(\mathbf{S})$. We shall refer to the members of $\mathcal{E} = \widehat{\mathrm{Sym}}(\mathbf{S})$ as *elasticity operators*, although this name should properly be reserved for the members of $\hat{\mathcal{P}}$.

The projection Splitsym(**M**) $\mapsto$ Sym(**S**) given by restricting a split symmetric operator on **M** to **S** has kernel Sym(**A**). From the defining equations above follows easily that $\mathcal{E}$ is complementary to Sym(**A**); thus restriction to **S** induces an isomorphism of $\mathcal{E}$ with $\mathcal{S}$. In particular the elasticity operators that restrict to automorphisms on **S** form a dense open set in $\mathcal{E}$.

The subset $\mathcal{P} \subset \mathcal{S}$ defined by the inequalities (3) consists of the positive definite symmetric operators on **S** and so is a convex open cone in $\mathcal{S}$. Consequently the twisted image $\mathcal{Q} = \hat{\mathcal{P}}$ is a convex open cone in $\mathcal{E} = \hat{\mathcal{S}}$. The members of the elasticity cone $\mathcal{Q}$ are split symmetric operators on **M**, but not necessarily positive definite, even when restricted to **S**. Nevertheless they posess a positivity property. For if $Y \in \mathbf{S}$ is of the form $y_{ij} = y_i y_j$ for a non-zero vector $\mathbf{y}$, then $Y'' = \hat{C}(Y)$ is a positive definite matrix in **S**, as is well known and easily checked. Therefore, if Y is a convex combination of these rank-1 matrices, $Y'' = \hat{C}(Y)$ is necessarily positive definite. But the convex hull of the matrices $y_{ij} = y_i y_j$ is the set $\mathbf{S}^+ \subset \mathbf{S}$ of positive semidefinite matrices. Consequently the members of $\mathcal{Q}$ carry the positive cone $\mathbf{S}^+$ into itself (and, apart from $Y = O$, even into the set of positive definite matrices).

3. THE ELASTICITY CONE. Let $(X_i)_{i=1\cdots r}$ and $(Z_i)_{i=1\cdots r}$ be any two r-tuples of linearly independent matrices from **M** (so $r \leq 9$), and define $T \in \mathrm{Lin}(\mathbf{M})$ by $T(Y) = \sum_i \langle Y, Z_i \rangle X_i$. Then T is a rank-$r$ operator on **M**, and any rank-r operator can be so represented. In fact we may suppose $X_1, \dots, X_r$ pairwise orthogonal.

If $Z_1 = X_1, \dots, Z_r = X_r$, then $T(Y) = \sum_i \langle Y, X_i \rangle X_i$ is a positive semidefinite, symmetric operator on $\mathbf{M}$, since

$$\langle T(Y), Y' \rangle \;=\; \sum_i \langle Y, X_i \rangle \langle Y', X_i \rangle$$

which is symmetric in Y, Y', and

$$\langle T(Y), Y \rangle \;=\; \sum_i \langle Y, X_i \rangle^2$$

which is non-negative. Conversely, any positive semidefinite, symmetric operator of rank r can be so represented, as follows from the spectral theorem.

The symmetric rank-1 operator

$$Y \mapsto \langle Y, X \rangle X \qquad\qquad (X \neq O)$$

will be denoted P_X. Of course P_X is simply $|X|^2$ times the orthogonal projection to $(\lambda X)_{\lambda \in \mathbf{R}}$. We wish to determine the effect of twisting on a symmetric operator. By the linearity of $\hat{}$ it suffices to determine $\hat{P}_X$ for any rank-1 operator P_X. But that's straightforward: We have $\langle E_{pq}, P_X(E_{rs}) \rangle \;=\; \langle E_{pq}, X \rangle \langle E_{rs}, X \rangle \;= x_{pq} x_{rs}$; i.e. if $P = P_X$, then $P_{pq,rs} = x_{pq} x_{rs}$. But then $\hat{P}_{pq,rs} = P_{pr,qs} = x_{pr} x_{qs}$ and so $Y'' = \hat{P}(Y)$ is given by $y''_{pq} = \sum_{r,s} x_{pr} x_{qs} y_{rs} = \sum_{r,s} x_{pr} y_{rs} x^t_{sq}$, i.e. $Y'' = XYX^t$.

Denote by Q_X the operator

$$Y \mapsto XYX^t \qquad\qquad (X \neq O)$$

on $\mathbf{M}$. We have just shown that P_X and Q_X correspond under $\hat{}$. In particular Q_X is split since P_X is symmetric. If X is a symmetric matrix, then P_X is also split and in fact belongs to $\mathcal{S}$, as is immediate from its coordinate expression. In this case therefore (and only this case) Q_X is an elasticity operator.

Now consider an operator $C \in \mathcal{P}$, i.e. a positive definite, symmetric operator on $\mathbf{S}$. By the spectral theorem we have $C = \lambda_1 P_{A_1} + \cdots + \lambda_6 P_{A_6}$, with $\lambda_1 \geq \cdots \geq \lambda_6 > 0$, say, (the eigenvalues of C) and $A_1, \ldots, A_6$ an orthogonal set of symmetric matrices of unit length (eigenvectors of C). Consequently the members of $\mathcal{Q}$ are precisely the operators $\hat{C} = \lambda_1 Q_{A_1} + \cdots + \lambda_6 Q_{A_6}$. But if λ is a positive scalar, then $\lambda P_A = P_X$ and $\lambda Q_A = Q_X$ with $X = \sqrt{\lambda} A$. In other words the elasticity cone $\mathcal{Q}$ consists of all operators

$$Q_{X_1} + \cdots + Q_{X_6}$$

where $(X_i)_{i=1,\cdots,6}$ varies over the orthogonal bases in $\mathbf{S}$.

4. AN EXAMPLE. The presentation above is quite useful. As an illustration consider the basis $X_1,\dots,X_6$ of $\mathbf{S}$ where $X_1 = E_{11}, X_2 = E_{22}, X_3 = E_{33}, X_4 = E_{23} + E_{32}, X_5 = E_{31} + E_{13}, X_6 = E_{12} + E_{21}$. This is an orthogonal basis for $\mathbf{S}$. Let $P_1,\dots,P_6$ be the corresponding projections and $Q_1,\dots,Q_6$ the corresponding elasticity operators. Set $C = P_1+\cdots+P_6$. Then $\hat{C} = Q_1+\cdots+Q_6$ and $\hat{C} \in \mathcal{Q}$. The action of $\hat{C}$ on a symmetric matrix $Y = (y_{ij})$ is given by $\hat{C}(Y) = Q_1(Y)+\cdots+Q_6(Y) = X_1YX_1 + \cdots + X_6YX_6$, or written out

$$\hat{C}(Y) = \begin{pmatrix} \operatorname{tr} Y & y_{12} & y_{13} \\ y_{21} & \operatorname{tr} Y & y_{23} \\ y_{31} & y_{32} & \operatorname{tr} Y \end{pmatrix}.$$

In particular the kernel of $\hat{C}$ is the space of traceless diagonal matrices in $\mathbf{S}$, a subspace of dimension 2. Thus $\hat{C}$ is only of rank 4 even though it sends all positive semidefinite matrices to positive definite matrices (O excepted).

The acoustic axes of C are easily computed. We find the three coordinate axes

$$\pm(1,0,0), \quad \pm(0,1,0), \quad \pm(0,0,1)$$

all associated to the triple eigenvalue 1, and the four diagonal axes

$$\pm\frac{1}{\sqrt{3}}(1,1,1), \quad \pm\frac{1}{\sqrt{3}}(-1,1,1), \quad \pm\frac{1}{\sqrt{3}}(1,-1,1), \quad \pm\frac{1}{\sqrt{3}}(1,1,-1)$$

all associated to the double eigenvalue $1/3$. Evidently the medium C is not generic since it has an odd number of acoustic axes, three of which are even of triple degeneracy and therefore necessarily unstable. On the other hand the four diagonal axes are in fact stable.

To see this we perceive the vectors $\mathbf{y} \in \mathbf{R}^3$ as column matrices and write the symmetric matrix (y_iy_j) as $\mathbf{y}\mathbf{y}^t$. Set $q(\mathbf{y}) = \mathbf{y}\mathbf{y}^t$; then q is a quadratic mapping into $\mathbf{S}$. The Christoffel matrix of C for the wave vector $\mathbf{y}$ is then $\hat{C}(q(\mathbf{y}))$. In other words we view the Christoffel mapping as the composite $\hat{C} \circ q$. Its derivative at $\mathbf{y}$ is then the composite linear mapping $\hat{C} \circ Dq(\mathbf{y})$. We compute the effect of $\hat{C} \circ Dq(y)$ on a tangent vector $\mathbf{v}$ to the unit sphere S^2 at $\mathbf{y}$ ($|\mathbf{y}| = 1$).

First represent $\mathbf{v}$ as the velocity vector at 0 of a parametrized path $\gamma(s)$ on S^2, so that $\gamma(0) = \mathbf{y}$, $\gamma'(0) = \mathbf{v}$. Then

$$Dq(\mathbf{y})\mathbf{v} = \frac{d}{ds}\bigg|_{s=0} q(\gamma(s)) = \gamma(0)\gamma'(0)^t + \gamma'(0)\gamma(0)^t = \mathbf{y}\mathbf{v}^t + \mathbf{v}\mathbf{y}^t$$

or, explicitely,

$$Dq(\mathbf{y})\mathbf{v} = \begin{pmatrix} 2y_1v_1 & y_1v_2 + y_2v_1 & y_1v_3 + y_3v_1 \\ y_1v_2 + y_2v_1 & 2y_2v_2 & y_2v_3 + y_3v_2 \\ y_1v_3 + y_3v_1 & y_2v_3 + y_3v_2 & 2y_3v_3 \end{pmatrix}$$

subject to the constraint $y_1v_1 + y_2v_2 + y_3v_3 = 0$ (since $\mathbf{v}$ is tangent to S^2 at $\mathbf{y}$). This holds at any unit vector $\mathbf{y}$ in $\mathbf{R}^3$. In particular at $\mathbf{y}_1 = 1/\sqrt{3}(1,1,1)$ we find

$$Dq(\mathbf{y}_1)\mathbf{v} = \frac{1}{\sqrt{3}} \begin{pmatrix} 2v_1 & v_1 + v_2 & v_1 + v_3 \\ v_1 + v_2 & 2v_2 & v_2 + v_3 \\ v_1 + v_3 & v_2 + v_3 & 2v_3 \end{pmatrix}$$

subject to the constraint $v_1 + v_2 + v_3 = 0$, and so

$$\hat{C}(Dq(\mathbf{y}_1)\mathbf{v}) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & v_1 + v_2 & v_1 + v_3 \\ v_1 + v_2 & 0 & v_2 + v_3 \\ v_1 + v_3 & v_2 + v_3 & 0 \end{pmatrix}.$$

As $\mathbf{v}$ runs through the tangent vectors of S^2 at $\mathbf{y}$ this matrix runs through a 2-dimensional linear subspace T_1 of $\mathbf{S}$ Locally around $Y_1'' = \hat{C}(q(\mathbf{y}_1))$ the variety Δ is smooth and can be expressed in terms of the Khatkevich equations, [3],

$$(u_{22} - u_{33})u_{12}u_{13} + u_{23}(u_{13}^2 - u_{12}^2) = 0$$
$$(u_{33} - u_{11})u_{12}u_{23} + u_{13}(u_{12}^2 - u_{23}^2) = 0,$$

where the u_{ij} are the standard coordinate functions on $\mathbf{S}$. Differentiating these equations at the matrix

$$Y_1'' = \begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 1 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix}$$

we obtain equations for the tangent space Δ_1 of Δ at Y_1'',

$$u_{22} - u_{33} + 2u_{13} - 2u_{12} = 0$$
$$u_{33} - u_{11} + 2u_{12} - 2u_{23} = 0.$$

Applying these to the image vectors $\hat{C}(Dq(\mathbf{y}_1)\mathbf{v})$ above we end up with the conditions

$$2(v_1 + v_3) - 2(v_1 + v_2) = 0$$
$$2(v_1 + v_2) - 2(v_2 + v_3) = 0,$$

which, together with the tangency condition $v_1+v_2+v_3=0$, shows that $\mathbf{v}=\mathbf{o}$. Thus T_1 and Δ_1 are complementary linear subspaces of $\mathbf{S}$ and in particular their sum is all of $\mathbf{S}$. This shows that $\hat{C}\circ q$ is transverse to Δ at $\pm\mathbf{y}_1$. Transversality at the other diagonal axes follows similarly.

5. RANK AND MULTIPLICITY. Let Σ be the subset of $\mathbf{S}$ of matrices of rank at most 2 and σ the subset of matrices of rank at most 1. Similarly let Δ be the subset of $\mathbf{S}$ of matrices with at most two different eigenvalues and δ the subset of matrices with one eigenvalue only. Finally let Σ^+, σ^+ and Δ^+, δ^+ be the corresponding subsets of positive semidefinite matrices.Then

$$\{O\}\subset\sigma\subset\Sigma\subset\mathbf{S}$$

and

$$\{O\}\subset\delta\subset\Delta\subset\mathbf{S}$$

Notice that δ consists of all scalar multiples of the unit matrix E and so is a line through the origin. The corresponding set σ is topologically more interesting. A symmetric matrix of rank ≤ 1 is of course semidefinite, so that $\sigma=\sigma^+\cup\sigma^-$ (where $^-$ means negative semidefinite). The sets $\sigma^\pm$ are precisely those that can be parametrized over $\mathbf{R}^3$ by

$$y_{ij}=\pm y_i y_j \tag{9}$$

Restriction of the parameters y_i to the unit sphere $|\mathbf{y}|=1$ in $\mathbf{R}^3$ yields two disjoint antipodal surfaces V^+ and V^- on the unit sphere $|Y|=1$ in $\mathbf{S}$. The mappings $\mathrm{S}^2\to V^\pm$ from the unit sphere defined by (12) are easily seen to be smooth double coverings. In fact V^+ is the classical realization of the projective plane $\mathrm{P}^2=\mathrm{P}(\mathbf{R}^3)$ as the Veronese surface in the 6-dimensional euclidean space $\mathbf{S}$. Thus σ is the cone through the origin in $\mathbf{S}$ that cuts the unit sphere along $V^+\cup V^-$. In particular σ is smooth except at the origin. From now on set $V^+=V$. Observe that $\pm\mathbf{y}$ is a stable acoustic axis for C if and only if $\hat{C}$ maps σ, or equivalently V, transversally to Δ at $Y=\mathbf{y}\mathbf{y}^t$.

We may express the multiple eigenvalue variety Δ in terms of σ and δ. First it is clear that Δ contains σ, since every Y of rank at most 1 has 0 as an eigenvalue of multiplicity at least 2 and so has at most 2 different eigenvalues. More generally every matrix of the form $Y+\lambda E$ with Y in σ has λ as an eigenvalue of multiplicity at least 2 and so belongs to Δ. Conversely every matrix in Δ can be written in this form. Thus Δ is the cylinder over σ with generator direction $\delta=(\lambda E)_{\lambda\in\mathbf{R}}$. Moreover it is a nondegenerate cylinder over σ since δ is not in the secant variety of σ. In particular Δ is a codimension 2 subvariety of $\mathbf{S}$, and δ is the singular locus of Δ. This implies (by definition) that a mapping into $\mathbf{S}$ which is transverse to Δ is transverse to $\Delta - \delta$ and δ. In particular, transverse mappings from V will avoid δ, since the dimension of V is less than the codimension of δ in $\mathbf{S}$.

It is already clear that the proper framework for our analysis is the projective formalism rather than the affine. Call a subset W of any vector space V *projective* if $W \neq \{O\}$ and $\lambda x \in W$ whenever $x \in W$, $\lambda \in \mathbf{R}$. Thus W is projective if it is made up entirely of directions, i.e. lines through the origin. Clearly the sets Σ, σ, and Δ, δ are projective subsets of $\mathbf{S}$. The projective set δ is a single direction in Δ, and Δ is the projective join of σ with δ (the union of all planes through δ from directions in σ), in traditional notation $\Delta = \sigma * \delta$.

6. THE GENERICITY SETS. The expression (7) defines by restriction a bilinear pairing $\hat{\mu}$ from $\mathcal{S} \times \mathbf{M}$ to $\mathbf{M}$, which is the composite of the twist with the standard action on $\mathbf{M}$, $\hat{\mu}(C,Y) = \hat{C}(Y)$. Equivalently, the standard pairing μ maps $\mathcal{E} \times \mathbf{M}$ bilinearly to $\mathbf{M}$. Since the members of $\mathcal{E}$ are split, it also maps $\mathcal{E} \times \mathbf{S}$ to $\mathbf{S}$ (as well as $\mathcal{E} \times \mathbf{A}$ to $\mathbf{A}$). Freezing the second variable at $Y \in \mathbf{S}$ yields the evaluation mapping $\mu_Y : \mathcal{E} \to \mathbf{S}$. This is an epimorphism for each Y except $Y = O$. Consequently we remove $O \in \mathbf{S}$ and end up with the pairing $\mathcal{E} \times \mathbf{S}^* \to \mathbf{S}$ (where * means "zero-matrix deleted"), which we continue to write μ. Pullback of the multiplicity filtration of $\mathbf{S}$ by μ gives the following situation

$$K^0 \subset K^\delta \subset K^\Delta \subset \mathcal{E} \times \mathbf{S}^* \xrightarrow{\mu} \mathbf{S} \supset \Delta \supset \delta \supset \{O\} \tag{10}$$

(so that K^Δ consists of all pairs (D,M) with $D \in \mathcal{E}$, $M \in \mathbf{S}^*$ and $D(M) \in \Delta$, etc). Clearly $K^0 \subset K^\delta \subset K^\Delta$ are subbundles of $\mathcal{E} \times \mathbf{S}^*$ (as spaces over $\mathbf{S}^*$), the first two even vector subbundles. In general if T is a vector subspace of $\mathbf{S}$ of codimension i, then $K^T = \mu^{-1}T$ is a vector subbundle of $\mathcal{E} \times \mathbf{S}^*$ of the same codimension i, and we have an exact sequence of vector bundles over $\mathbf{S}^*$

$$0 \longrightarrow K^0 \longrightarrow K^T \xrightarrow{\bar{\mu}} T \times \mathbf{S}^* \longrightarrow 0 \tag{11}$$

where $\bar{\mu}$ is the bundle map $\bar{\mu}(C,M) = (\mu(C,M),M)$. Since Δ is generated by directions l, K^Δ is a pencil of vector bundles K^l of fiber dimension 16 with axis K^0 of fiber dimension 15. Finally K^Δ itself is a projective bundle of fiber dimension 19.

Of course all this remains true by restriction to any subset in the base $\mathbf{S}^*$. In particular restriction to V yields the situation

$$K_V^0 \subset K_V^\delta \subset K_V^\Delta \subset \mathcal{E} \times V \xrightarrow{\mu} \mathbf{S} \supset \Delta \supset \delta \supset \{O\} \tag{12}$$

where K_V^Δ is the set of pairs (C,Y) with $Y \in V$ and $C(Y) \in \Delta$, etc.

We use this to define the *unstable variety* $W \subset K_V^{\Delta}$ as the set of pairs (D, Y) such that $D|V$ is tangent at Y to Δ. Here *tangent* means non-transversal. Since Δ is smooth outside δ only, this means that either DY is in $\Delta - \delta$ and $D|V$ is tangent to $\Delta - \delta$ at Y, or DY is in δ and $D|V$ is tangent to δ at Y. But the last condition simply means that DY is in δ, i.e. that (D, Y) is in K^{δ}, since $D|V$ is tangent to δ at any Y such that $DY \in \delta$. Consequently we have $K_V^{\delta} \subset W \subset K_V^{\Delta}$.

Next, set $\hat{\mathcal{D}} = pr_1(W)$ and observe that W is the set of critical points and $\hat{\mathcal{D}}$ the set of critical values (the *discriminant* set) of the mapping $pr_1|(K_V^{\Delta} - K_V^{\delta}) \cup K_V^{\delta}$. Clearly W is a closed algebraic subvariety of K_V^{Δ} and pr_1 a proper mapping, and so $\hat{\mathcal{D}}$ is a closed semialgebraic subset of $\mathcal{E}$. By Sard's theorem [4] $\hat{\mathcal{D}}$ is nowhere dense and so $\dim \hat{\mathcal{D}} \leq 20$. On the other hand $\hat{\mathcal{D}}$ disconnects $\mathcal{E}$ and so $\dim \hat{\mathcal{D}} = 20$.

If C is a generic medium, i.e. $\hat{C} \in \mathcal{E} - \hat{\mathcal{D}}$, then $\hat{C}$ maps V into $\mathbf{S}$ transversally to Δ. In particular $\hat{C}$ hits $\Delta - \delta$ for a finite number of matrices $Y = (y_i y_j)$ only and δ not at all. The corresponding directions $\pm\mathbf{y}$ in $\mathbf{R}^3$ are the accoustic axes of C.

LEMMA. *Let C be a generic elastic medium with p accoustic axes. There is a neighbourhood $\mathcal{U}$ of C in $\mathcal{S}$ such that every $T \in \mathcal{U}$ is generic with p accoustic axes.*

Proof. If $p = 0$, we have a medium without accoustic axes. Then $\hat{C}$ maps V to a compact set in $\mathbf{S}$ bounded away from Δ. No small perturbation of C can change this.

In the case $p \neq 0$ consider the mapping $\pi : K_V^{\Delta} \to \mathcal{E}$, the restriction of pr_1 to K_V^{Δ}. The fact that $\hat{C}$ maps V transversally to Δ is equivalent to $\hat{C}$ being a regular value of of π. (The transversality lemma, [4].) Let $(\hat{C}, Y_1), (\hat{C}, Y_2), \cdots, (\hat{C}, Y_p) \in K_V^{\Delta}$ be the corresponding regular points (so that $Y_1, Y_2, \cdots, Y_p$ represents the accoustic axes of C). Let $U_1, U_2, \cdots, U_p$ be pairwise disjoint neighbourhoods of $(\hat{C}, Y_1), (\hat{C}, Y_2), \cdots, (\hat{C}, Y_p)$ in K_V^{Δ}, which are also disjoint from W, and so small that they are mapped bianalytically onto neighbourhoods $\hat{\mathcal{U}}_1, \hat{\mathcal{U}}_2, \cdots, \hat{\mathcal{U}}_p$ of $\hat{C}$ by π. (The inverse function theorem.) Set $\hat{\mathcal{U}} = \cap_i \hat{\mathcal{U}}_i - \pi(K_V^{\Delta} - \cup_i U_i)$. Then each point of $\hat{\mathcal{U}}$ is the projection of precisely p regular pairs, one from each U_i. This defines $\hat{\mathcal{U}}$ and therefore $\mathcal{U}$.

The lemma shows that the function

$$C \mapsto \text{ number of accoustic axes of } C$$

is locally constant on $\mathcal{E} - \hat{\mathcal{D}}$ and consequently constant on each connected component of $\mathcal{E} - \hat{\mathcal{D}}$.

Let us estimate the actual number of accoustic axes of a generic medium C. We consider $\hat{C}$ on the full projectiv set σ rather than on the partial cut V only. The situation is the following,

$$\sigma \subset \mathbf{S} \xrightarrow{\hat{C}} \mathbf{S} \supset \Delta$$

and we need to know the number of axes $[Y]$ in σ such that $\hat{C}(Y) \in \Delta$. Since a small perturbation of C will not change the number of axes (by the lemma above), we may if necessary perturb C to ensure that $\hat{C}$ is invertible on $\mathcal{S}$ (cf. last part of section 2). Now complexify everything in sight:

$$\sigma_{\mathbf{C}} \subset \mathbf{S}_{\mathbf{C}} \xrightarrow{\hat{C}} \mathbf{S}_{\mathbf{C}} \supset \Delta_{\mathbf{C}}.$$

In the case of Δ it is important to take the proper complexification $\Delta_{\mathbf{C}} = \sigma_{\mathbf{C}} * \delta_{\mathbf{C}}$, where $\Delta = \sigma * \delta$ is the realization of Δ as the projective join of σ with δ (the cylinder on σ along δ). Then $\deg \Delta_{\mathbf{C}} = \deg \sigma_{\mathbf{C}} = 4$. Consequently the solution set $\sigma_{\mathbf{C}} \cap (\hat{C})^{-1}\Delta_{\mathbf{C}}$ contains 16 axes when counted with multiplicities (Bezout's theorem). Moreover, whenever we have a complex axis $[Y]$, we also have its complex conjugate $[\bar{Y}]$. Thus, of the 16 axes there is an even number of *real* solutions. Finally, since (the complexified) $\hat{C}$ is transverse to $\Delta_{\mathbf{C}}$ along $\sigma \subset \sigma_{\mathbf{C}}$, all real solutions are *simple*. This concludes the statements made earlier.

REMARK. To elicit further geometric properties of the material C we ought to pay attention to the behaviour of Y'' when $\mathbf{y}$ varies more generally in complex space $\mathbf{C}^3$, and $\mathbf{S}$ and Δ are replaced with their complexifications $\mathbf{S}_{\mathbf{C}}$ and $\Delta_{\mathbf{C}}$. In fact in the complex situation there is even an additional critical set $\Gamma_{\mathbf{C}} \subset \mathbf{S}_{\mathbf{C}}$, given by the dicriminant equation $d(Y) = 0$, cf. [3]. $\Gamma_{\mathbf{C}}$ is the hypersurface of complex symmetric matrices with eigenvalues of multiplicity ≥ 2, while $\Delta_{\mathbf{C}} \subset \Gamma_{\mathbf{C}}$ is the subset of matrices with eigenspaces of dimension ≥ 2. These two varieties have the same real part $\Gamma = \Delta$ (so both are complexifications of Δ). Moreover $\Delta_{\mathbf{C}}$ is the singularity subset of $\Gamma_{\mathbf{C}}$.

If the Christoffel mapping is transverse to $\Delta_{\mathbf{C}}$ (say) along $\mathbf{R}^3 \subset \mathbf{C}^3$, then it is also transverse along the imaginary part $i\mathbf{R}^3$; the two domains yield corresponding positive- and negative definite matrices only. Of interest, however, is the behaviour of Y'' along the null cone F of complex directions $[\mathbf{z}]$ where $Y''(\mathbf{z})$ has 0 as eigenvalue. The null cone enters in connection with the limit case $\omega = 0$ of (4), which is a solution of the corresponding Laplace system

$$\sum_{q,r,s} C_{pq,rs} \frac{\partial^2 u_r}{\partial x_q \partial x_s} = 0. \qquad (p = 1, 2, 3)$$

In this case the parameters $\mathbf{a}$ and $\mathbf{z}$ satisfy the equation $Y''(\mathbf{z})\mathbf{a} = \mathbf{o}$.

The null-cone F is a complex surface of degree 6 in $\mathbf{C}^3$, invariant under complex conjugation, but without real directions. However, through every direction in F passes a unique real *plane*; its coefficients are real, determined up to a real scalar, and so determines a direction in $\mathbf{R}^3$. Special directions (e.g. representing the $[\mathbf{z}]$ in F such that $Y''(\mathbf{z})$ lies in $\Delta_{\mathbf{C}}$), now occur as in the dynamic situation. A study of these special directions —called elastic axes— was initiated recently by Lothe in a different (analytic) setting. The present geometric point of view has the advantage of being directly invariant. However, we will not discuss elastic axes in this paper. The purpose of this remark is only to point out that a proper definition of of generic media should require stability of all exeptional axes, elastic as well as acoustic.

References

1. Alshits, V. I. and Lothe, J., *Elastic waves in triclinic crystals*, Kristallografiya **24** (1979), 672–693.
2. Chadwick, P. and Smith, G. D., *Foundations of the theory of surface waves in anisotropic elastic materials*, Advances in Applied Mathematics **17** (1977), 303–376.
3. Holm, P. and Lothe, J., *The topological nature of the polarisation field for body waves in anisotropic elastic media*, Proc. R. Soc. Lond. **A 370** (1980), 331–350.
4. Guillemin, V. and Pollack, A., *Differential Topology*, Prentice Hall, 1974.

Per Holm, University of Oslo, Institute of Mathematics, P.O.Box 1053 Blindern, 0316 OSLO 3, NORWAY. E-mail: per@math.uio.no